Int. J. Solids Structures, 1977, Vol. 13, pp. 221–228. Pergamon Press. Printed in Great Britain

# SPREAD OF PLASTICITY FROM STACKED STRESS CONCENTRATIONS

B. L. KARIHALOO†

Department of Solid Mechanics, The Technical University of Denmark, Lyngby, Denmark

(*Received* 23 *April* 1976; *revised* 21 *July* 1976)

**Abstract**—A large elastic solid containing an infinite sequence of slitlike relaxed cracks with a constant distance of vertical separation is considered. The solid is deforming under plane strain shear conditions (mode II). The plastic relaxation around each of the cracks is represented by a suitable distribution of edge dislocations coplanar with the crack itself, the distribution being determined from a singular integral equation. This equation is solved numerically using an expansion of the non-singular part of the kernel in a series of Chebyshev polynomials. Solutions are obtained for the extent of spread of plasticity from each crack and for the associated dislocation distribution as a function of the crack spacing and the applied load. The results are applied to a brief discussion of the fracture process at stress concentrations using the crack opening displacement criterion.

## INTRODUCTION

A common feature of most natural and manufactured solid materials is the presence in them of randomly oriented and distributed cracks of various types and scales. These cracks (inhomogeneities) can significantly alter the gross elastic response of a solid through an interaction of their stress fields, especially if they are separated by less than several crack lengths. Similarly, the relaxation of stresses round the inhomogeneities can distinctly change the fracture characteristics of the solid. However, an exact analytical investigation is almost impossible owing to the randomness of the orientation and of the distribution of cracks. Nevertheless, a fairly accurate estimate of the change in the elastic response of the solid with inhomogeneities and in the relaxation of stresses round the latter can be obtained by studying an infinite homogeneous and isotropic elastic solid containing a regular distribution of cracks.

Thus, Louat[1] treated the case of a solid in anti-plane strain state with an infinite sequence ("stack") of un-relaxed cracks, while Smith[2] gave exact solutions for the extent of spread of plasticity from each crack. The model of an infinite solid with an isolated relaxed crack was considered by, among others, Bilby, Cottrell and Swinden[3], Dugdale[4], Rice[5], Kostrov and Nikitin[6], Karihaloo[7]. Koiter[8] studied the problem of an infinite solid in in-plane shear state with a single periodic stack of un-relaxed cracks, while Smith[2] gave approximate solutions for the spread of plasticity from each crack for the less interesting case of widely spaced cracks. Again, Koiter[9] and, later, Paris and Sih[10] solved the problem of a solid subjected to in-plane shear stress and containing an infinite row of collinear unrelaxed cracks; an exact solution of the corresponding problem of relaxed cracks was given by Bilby, Cottrell, Smith and Swinden[11]. Here, it is worth mentioning that, no matter whether the body containing an infinite row of collinear relaxed cracks deforms in antiplane strain mode (mode III) or under plane strain shear conditions (mode II), the results for the extent of spread of plasticity will be identical and the corresponding dislocation distribution functions, representing the slitlike cracks, will differ only by a constant factor involving Poisson's ratio.

The problem of a solid subjected to tensile loading (mode I) and containing an infinite row of collinear unrelaxed cracks was studied by Benthem and Koiter[12]. A more difficult problem of an infinite solid containing a doubly periodic rectangular array of unrelaxed slitlike cracks was examined in detail for all the three modes of loading by Delameter, Herrmann and Barnett[13] and Delameter and Herrmann[14]. These authors based their study on the equivalence of slitlike cracks and suitable distributions of straight dislocations, discussed in detail by Bilby and Eshelby[15]. This seems to be the only feasible method, because the mapping technique normally used in solving the simplest crack problems presents formidable mathematical difficulties.

†Present address: Department of Civil Engineering, The University of Newcastle, N.S.W. 2308, Australia.

The aim of the present study is to extend the dislocation modelling technique to the investigation of stress relaxation round inhomogeneities that are not coplanar. Thus, section 1 considers the in-plane shear model of the spread of plasticity from an infinite sequence of cracks $|x| \leq c$, $y = \pm nh$ ($n = 0, 1, 2$, etc.) in an infinite elastic solid subject to an applied shear stress $\sigma_{xy} = \sigma$ at infinity; the singular integral equation governing the problem cannot be readily solved in an exact form. Instead, an efficient approximate procedure is presented which uses an expansion of the non-singular part of the kernel in a series of Chebyshev polynomials (Section 2). The models of an infinite sequence of non-coplanar unrelaxed cracks (Koiter[8]), an isolated relaxed crack (Bilby *et al.*[3]) and Smith's[2] model for widely spaced cracks deforming under in-plane shear conditions, form particular cases of the present study. In Section 3 the in-plane shear results are used to discuss briefly some applications to the problem of fracture at stress concentrations, in particular, when the cracks are closely spaced.

## 1. THE IN-PLANE SHEAR MODEL (MODE II)

Let us consider the spread of plasticity from an infinite sequence of cracks $|x| \leq c$, $y = \pm nh$ in an infinite, isotropically elastic solid deforming in a plane strain mode (sliding mode II) under an externally applied shear stress $\sigma_{xy} = \sigma$, the displacement discontinuity across each crack being in the $x$-direction. This discontinuity along each of the planes $y = \pm nh$ can be represented by a continuous distribution of long edge dislocations parallel to the $x$-axis and lying in the planes $y = \pm nh$. Plastic relaxation around the tips of the cracks is represented by edge dislocations coplanar with the cracks, the resistance to motion of these dislocations being due to a friction stress (taken to be equal to the yield stress of the material) $\sigma_y > \sigma$, and not zero as for the dislocations which represent the freely slipping cracks. If the plastic zones spread out to a distance $a$, positive and negative edge dislocations (positive edges have their extra half-plane of atoms in the positive $y$-direction) lie respectively in the intervals $0 \leq x \leq a$, $y = \pm nh$ and $-a \leq x \leq 0$, $y = \pm nh$. Those in the plastically relaxed regions ahead of the cracks $c < |x| \leq a$, $y = \pm nh$ are subject to a net applied stress $\sigma_{xy} = \sigma - \sigma_y$, in addition to the interaction stresses from all the other dislocations, whilst those in the ranges $|x| < c$, $y = \pm nh$ are subject to a stress $\sigma_{xy} = \sigma$, besides the various interaction stresses.

It is evident that the distribution of dislocations will be the same in each of the planes $y = \pm nh$. Thus we suppose that there are $f(x)\delta x$ dislocations each of Burgers vector $b > 0$ in any interval $\delta x$. We are required to determine $f(x)$ and the relation between $c$ and $a$ for various values of the physical parameters $\sigma_y$ and $\sigma$ as a function of the crack spacing $h$.

In order to calculate the interaction stresses the problem can be considered as one involving the interaction between vertical arrays of edge dislocations. The shear stress $\sigma_{xy}$ (the normal stress $\sigma_{yy} = 0$ due to symmetry, so that the crack is freely slipping) due to such an array of positive edge dislocations situated in the plane $x = x'$ is

$$\sigma_{xy} = \frac{\mu b}{2\pi(1-\nu)} \sum_{n=-\infty}^{\infty} \frac{(x-x')[(x-x')^2 - n^2h^2]}{[(x-x')^2 + n^2h^2]^2}, \tag{1}$$

at a point $x$ along one of the planes $y = \pm nh$, where $\mu$ is the shear modulus, $b$ the Burgers vector of each edge dislocation, and $\nu$ Poisson's ratio.

The infinite sum over $n$ can be evaluated in terms of hyperbolic sine, and (1) may be rewritten as

$$\sigma_{xy} = \frac{\mu b}{2\pi(1-\nu)} \frac{1}{(x-x')} \left[\frac{\pi(x-x')/h}{\sinh \pi(x-x')/h}\right]^2. \tag{2}$$

Hence, the integral equation, which expresses the requirement that the resultant shear stress on any dislocation in the distribution vanish when the system is in equilibrium, can be obtained by summing the effects due to all the other dislocations, and is

$$\int_{-a}^{a} \frac{\mu b}{2\pi(1-\nu)} \frac{1}{(x-x')} \left[\frac{\pi(x-x')/h}{\sinh \pi(x-x')/h}\right]^2 f(x')\,dx' + \tau(x) = 0 \tag{3}$$

for $|x| \leq a$, where $\tau(x)$ is the applied stress at $x$ and it is understood that the Cauchy principal value of the singular integral is used. This singular integral equation, in contrast to the corresponding equation for anti-plane strain model, does not seem to have a closed form solution for the distribution function $f(x')$. An iterative solution was presented by Smith[2] for the less interesting case of widely spaced cracks. However, from a practical point of view, the more important case is when the cracks are close together. We present in the next section an efficient approximate method of solution (using orthogonal polynomial expansions) that is uniformly valid for all values of crack spacing. Before that we modify eqn (3) and introduce non-dimensional variables.

Let $x_1 = x/a$, $x_1' = x'/a$, $h_1 = h/\pi c$, $f(x_1') = \mu b f(x')/2\pi(1-\nu)\sigma_y$ and denote $\alpha = c/a$ ($\leq 1$, equality holding for unrelaxed cracks). Then eqn (3) takes the following form (subscript 1 has been omitted)

$$\int_{-1}^{1} \frac{1}{(x-x')}\left[\frac{(x-x')/\alpha h}{\sinh (x-x')/\alpha h}\right]^2 f(x')\,dx' + P(x) = 0, \tag{4}$$

where $P(x) = \sigma/\sigma_y$ in the interval $0 \leq |x| \leq \alpha$ and $P(x) = \sigma/\sigma_y - 1$ in the plastic regions ahead of the crack tips $\alpha < |x| \leq 1$.

Furthermore, let us rewrite (4) as follows

$$\int_{-1}^{1} f(x')\left\{\frac{1}{x-x'} + K(x',x)\right\} dx' + P(x) = 0, \tag{5}$$

where the non-singular part $K(x',x)$ of the kernel is given by

$$K(x',x) = \frac{1}{(x-x')}\left\{\left(\frac{(x-x')/\alpha h}{\sinh (x-x')/\alpha h}\right)^2 - 1\right\}. \tag{6}$$

It may be noted that, as $x \to x'$, $K(x',x) \to 0$. It is also interesting to note that $K(x',x)$ represents the additional stress due to the interaction of cracks. In fact, for an isolated relaxed crack, (5) coincides with the corresponding singular integral equation of Bilby *et al.*[3], when $K(x',x) = 0$.

## 2. METHOD OF SOLUTION

As mentioned above, the singular integral eqn (5) does not seem to have a closed-form solution. We, therefore employ an approximate method of solution, similar to that used in Refs. [13, 14]. This method proves to be quite efficient for obtaining accurate numerical results.

For given values of $h$ and $\sigma/\sigma_y$, the non-singular part of the kernel $K(x',x)$ is known. (It should be mentioned that $\alpha$ and $\sigma/\sigma_y$ are related through the necessary condition for the existence of a solution to the singular integral eqn (5), see below). It may thus be expanded in a series of orthogonal polynomials in the variable $x$, the coefficients of the series being functions of $x'$. Chebyshev polynomials seem to be eminently suitable for our purpose, as will become clear later. We therefore assume that

$$K(x',x) = \sum_{n=0}^{\infty} A_n(x')T_n(x), \tag{7}$$

where $T_n(x)$ is the $n$th Chebyshev polynomial of the first kind, defined by

$$T_n(x) = \cos(n \cos^{-1} x),$$

with $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, and so on. From the orthogonality property of $T_n(x)$, viz,

$$\int_{-1}^{1} \frac{\{T_n(x)\}^2}{\sqrt{1-x^2}}\,dx = \begin{cases} \pi, & \text{if } n = 0 \\ \pi/2, & \text{if } n = 1, 2, 3, \ldots \end{cases}$$

it follows that

$$A_0(x') = \frac{1}{\pi}\int_{-1}^{1} \frac{K(x', x)}{\sqrt{1-x^2}}\,dx$$

and

$$A_n(x') = \frac{2}{\pi}\int_{-1}^{1} \frac{K(x', x)T_n(x)}{\sqrt{1-x^2}}\,dx; \quad n = 1, 2, 3, \ldots \tag{8}$$

Substituting (7) in (5), we get

$$\int_{-1}^{1} \frac{f(x')\,dx'}{(x-x')} + \sum_{n=0}^{\infty} T_n(x)\int_{-1}^{1} f(x')A_n(x')\,dx' + P(x) = 0, \tag{9}$$

which may be rewritten in a concise form as

$$\int_{-1}^{1} \frac{f(x')}{(x-x')}\,dx' = \sum_{n=0}^{\infty} a_n T_n(x) - P(x), \tag{10}$$

where

$$a_n = -\int_{-1}^{1} f(x')A_n(x')\,dx'. \tag{11}$$

The condition for the existence of a solution of the singular integral eqn (10) is [16]

$$\sum_{n=0}^{\infty} a_n \int_{-1}^{1} \frac{T_n(x)}{\sqrt{1-x^2}}\,dx - \int_{-1}^{1} \frac{P(x)}{\sqrt{1-x^2}}\,dx = 0, \tag{12}$$

$f(x)$ then vanishing at the tips of the plastic zones ($x = \pm 1$) ahead of the cracks. Equation (12) specifies the distance to which the dislocations (yield zones) spread under a given applied stress $\sigma$, i.e. it gives a relation between $\alpha$ and $\sigma/\sigma_y$. In fact, as a consequence of the orthogonality properties

$$\int_{-1}^{1} \frac{T_0^2(x)}{\sqrt{1-x^2}}\,dx = \pi$$

and

$$\int_{-1}^{1} \frac{T_0(x)T_n(x)}{\sqrt{1-x^2}}\,dx = 0, \quad n = 1, 2, 3, \ldots,$$

eqn (12) reduces to

$$\pi a_0 - \int_{-1}^{1} \frac{P(x)}{\sqrt{1-x^2}}\,dx = 0,$$

whence it follows that

$$\cos^{-1}\alpha = \frac{\pi}{2}\left(\frac{\sigma}{\sigma_y} - a_0\right). \tag{13}$$

The corresponding expression for an isolated relaxed crack is

$$\cos^{-1}\alpha = \frac{\pi}{2}\frac{\sigma}{\sigma_y}$$

A comparison of these two expressions easily shows that the additional term $(\pi/2)a_0$ is a manifestation of the presence of an infinite sequence of relaxed cracks.

When the existence condition (13) is satisfied, eqn (10) gives the distribution function

$$f(x)=\frac{\sqrt{1-x^2}}{\pi^2}\sum_{n=0}^{\infty}a_n\int_{-1}^{1}\frac{T_n(\lambda)}{\sqrt{1-\lambda^2}(\lambda-x)}\mathrm{d}\lambda$$
$$-\frac{\sqrt{1-x^2}}{\pi^2}\int_{-1}^{1}\frac{P(\lambda)}{\sqrt{1-\lambda^2}(\lambda-x)}\mathrm{d}\lambda. \tag{14}$$

Integrating the second term on the right-hand side of (14), we get

$$\eta(x)=-\frac{\sqrt{1-x^2}}{\pi^2}\int_{-1}^{1}\frac{P(\lambda)}{\sqrt{1-\lambda^2}(\lambda-x)}\mathrm{d}\lambda$$
$$=\frac{1}{\pi^2}\left\{\cosh^{-1}\left(\left|\frac{1-\alpha x}{\alpha-x}\right|\right)-\cosh^{-1}\left(\left|\frac{1+\alpha x}{\alpha+x}\right|\right)\right\}.$$

Furthermore, let

$$\psi_n(x)=\frac{\sqrt{1-x^2}}{\pi^2}\int_{-1}^{1}\frac{T_n(\lambda)}{\sqrt{1-\lambda^2}(\lambda-x)}\mathrm{d}\lambda,\quad n=0,1,2,\ldots.$$

Equation (14) then takes the form

$$f(x)=\sum_{n=0}^{\infty}a_n\psi_n(x)+\eta(x). \tag{15}$$

It may be noted that $\eta(x)$ is the solution for an isolated relaxed crack. The $a_n$ s are as yet unknown. However, before proceeding with their determination it is worthwhile simplifying the functions $\psi_n$. It is easily shown that

$$\psi_0(x)=\frac{\sqrt{1-x^2}}{\pi^2}\int_{-1}^{1}\frac{\mathrm{d}\lambda}{\sqrt{1-\lambda^2}(\lambda-x)}$$
$$=0.$$

Furthermore, as a consequence of the identity

$$\int_{-1}^{1}\frac{T_{n+1}(\lambda)}{\sqrt{1-\lambda^2}(\lambda-x)}\mathrm{d}\lambda=\pi U_n(x),\quad n=0,1,2,\ldots$$

we have

$$\psi_{n+1}(x)=\frac{\sqrt{1-x^2}}{\pi}U_n(x),\quad n=0,1,2,\ldots.$$

where $U_n(x)$ is the $n$th Chebyshev polynomial of the second kind, defined by

$$U_n(x)=\frac{\sin[(n+1)\cos^{-1}x]}{\sin(\cos^{-1}x)},\quad n=0,1,2,\ldots,$$

with $U_0(x)=1$, $U_1(x)=2x$, $U_2(x)=4x^2-1$, and so on. It should be noted that $\eta(x)$ is an odd function. In fact, it can be shown that the solution $f(x)$ of the singular integral eqn (10) is an odd function. Consequently, given that $\psi_0(x)=0$, (15) may be rewritten as

$$f(x)=\sum_{n=1}^{\infty}b_n\psi_{2n}(x)+\eta(x),$$

where $\psi_{2n}(x) = (\sqrt{1-x^2}/\pi)U_{2n-1}(x)$ is an odd function, and

$$b_n = -\int_{-1}^{1} f(x')A_{2n}(x')\,dx'.$$

The $b_n$'s which are as yet unknown are determined by substituting $f(x)$ into the expression for $b_n$. This leads to an infinite system of linear algebraic equations

$$\sum_{j=1}^{\infty} C_{ij}b_j = D_i, \qquad i = 0, 1, 2, \ldots, \tag{16}$$

where

$$D_i = -\int_{-1}^{1} \eta(x')A_{2i}(x')\,dx'$$

and

$$C_{ij} = \delta_{ij} + \int_{-1}^{1} \psi_{2j}(x')A_{2i}(x')\,dx', \tag{17}$$

$\delta_{ij}$ being the Kronecker delta. Having determined the coefficients $b_n$ and, hence, the function $f(x)$, the original coefficient $a_0$ entering (13) is evaluated from

$$a_0 = -\int_{-1}^{1} f(x)A_0(x)\,dx.$$

For future use, we calculate the number of dislocations in any given interval and the relative displacement $\Delta(x)$ of the positive side of slip plane with respect to the negative side.

In this connection we integrate the dislocation distribution function between 0 and $x$ and find the number of dislocations $N(x)$ to be

$$\begin{aligned} N(x) &= \sum_{n=1}^{\infty} b_n \int_0^x \psi_{2n}(x)\,dx + \int_0^x \eta(x)\,dx \\ &= \sum_{n=1}^{\infty} b_n \int_0^x \psi_{2n}(x)\,dx + \frac{1}{\pi^2}(x-\alpha)\cosh^{-1}\left(\left|\frac{1-\alpha x}{\alpha - x}\right|\right) \\ &\quad - \frac{1}{\pi^2}(x+\alpha)\cosh^{-1}\left(\left|\frac{1+\alpha x}{\alpha + x}\right|\right) + 2\frac{1}{\pi^2}\alpha\cosh^{-1}\left(\frac{1}{\alpha}\right). \end{aligned} \tag{18}$$

In particular,

$$N(1) = \sum_{n=1}^{\infty} b_n \int_0^1 \psi_{2n}(x)\,dx + \frac{2}{\pi^2}\alpha\cosh^{-1}\left(\frac{1}{\alpha}\right)$$

and

$$N(\alpha) = \sum_{n=1}^{\infty} b_n \int_0^{\alpha} \psi_{2n}(x)\,dx + \frac{2}{\pi^2}\alpha\left\{\cosh^{-1}\left(\frac{1}{\alpha}\right) - \cosh^{-1}\left(\frac{\alpha^2+1}{2\alpha}\right)\right\}.$$

The relative displacement $\Delta(x)$ is

$$\Delta(x) = b\{N(1) - N(X)\},$$

whence it follows that

$$\Delta^*(x) = \frac{\pi^2}{2\alpha}\sum_{n=1}^{\infty} b_n \int_x^1 \psi_{2n}(x)\,dx + \frac{(x+\alpha)}{2\alpha}\cosh^{-1}\left\{\left|\frac{1+\alpha x}{\alpha + x}\right|\right\} - \frac{(x-\alpha)}{2\alpha}\cosh^{-1}\left\{\left|\frac{1-\alpha x}{\alpha - x}\right|\right\}, \tag{19}$$

where the non-dimensional $\Delta^*(x) = \Delta(x)\pi\mu/4c(1-\nu)\sigma_y$.

Finally, the relative displacement at $x = \alpha$ (i.e. at crack tip) is given by

$$\Delta^*(\alpha) = \frac{\pi^2}{2\alpha} \sum_{n=1}^{\infty} b_n \int_{\alpha}^{1} \psi_{2n}(x)\, dx + \ln(1/\alpha). \tag{20}$$

The expression for an isolated relaxed crack is easily recovered by setting the first term on the right hand side of (20) equal to zero.

The coefficients $b_n$ were determined for given values of $\alpha$ and $h$ from the linear system of eqns (16). The system was truncated at $i = j = 8$, which assured sufficient accuracy for all the cases treated. Integrals were evaluated by Simpson's rule, after making a change of variables, where necessary, to render the integrands non-singular. Having evaluated the function $f(x)$ and, hence, the coefficient $a_0$, the corresponding value of $\sigma/\sigma_y$ was evaluated from (13).

The extent of the spread of plasticity as a function of the applied stress and for various values of crack spacing is shown in Fig. 1. For the sake of comparison, the graph for an isolated relaxed crack ($h \to \infty$) is also reproduced. The results for widely spaced cracks are in good agreement with those obtained by Smith[2] by a perturbation technique.

## 3. DISCUSSION

As is evident from the figure, for a given value of the extent of the spread of plasticity, the required shear stress $\sigma$ decreases with decreasing distance between the cracks. For a better appreciation of this fact it is worthwhile considering the behaviour of unrelaxed cracks.

The number $N$ of edge dislocations in each half of one of the planes $y = \pm nh$ is given by (18). Since $\psi_n$ increases with a reduction of $h$, it follows that $N$ increases as the cracks come closer. On the other hand, since $N$ is a measure of the crack opening displacement, it follows that the effective stress-intensity factor increases with a reduction of the crack spacing ($h$). This is in complete agreement with the results reported in [13, 14]. It is also interesting to note that this conclusion is the opposite of the one for anti-plane shear model, which emphasizes the need for exercising caution in using anti-plane strain model for predicting the behaviour of the solid in the more frequent plane strain conditions encountered in practice.

In order to apply the results of the present study to the problem of fracture at stress concentrations we employ the crack opening displacement criterion, according to which fracture is initiated at a crack when the relative displacement there $\Delta(c)$ exceeds some critical value $\Delta_{crit}$, assumed to be a property of the material. Now, as, for a given value of the applied stress $\sigma$, $a$ increases as the distance between the cracks decreases (Fig. 1), (20) implies that the tendency towards fracture increases as the cracks come closer, which is just the opposite of what is expected for an anti-plane shear model.

The above argument serves to indicate how the presence of a number of inhomogeneities adversely alters the fracture characteristics of a material by lowering the fracture stress in comparision with that for an isolated crack.

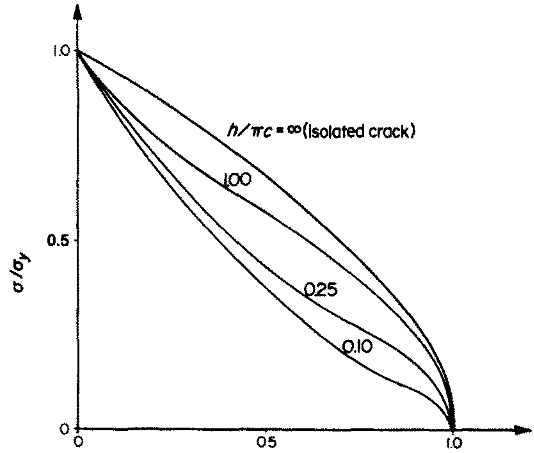


Fig. 1. The extent of spread of plasticity $a$ from an infinite sequence of cracks in an infinite solid deforming under plane strain conditions due to $\sigma_{xy} = \sigma$, for different values of crack spacing $h$.

## REFERENCES

1. N. Louat, *Phil. Mag.* **8**, 1219 (1962).
2. E. Smith, *Proc. Roy. Soc. Lond.* **A282**, 422 (1964).
3. B. A. Bilby, A. H. Cottrell and K. H. Swinden, *Proc. Roy. Soc. Lond.* **A272**, 304 (1963).
4. D. S. Dugdale, *J. Mech. Phys. Solids* **8**, 100 (1960).
5. J. R. Rice, *J. Mech. Phys. Solids* **22**, 17 (1974).
6. B. V. Kostrov and L. V. Nikitin, *Appl. Math. Mech.* **31**, 334 (1967); (Translation of *Prikladnaya Matematika i Mekhanika, PMM*).
7. B. L. Karihaloo, To be published.
8. W. T. Koiter, *Problems in Continuum Mechanics*, p. 246. Society for Industrial and Applied Mathematics, Philadelphia (1961).
9. W. T. Koiter, *Ingenieur-Archiv.* **28**, 163 (1959).
10. P. C. Paris and G. C. Sih, *STP* 381. p. 30. American Society for Testing Materials, Philadelphia (1965).
11. B. A. Bilby, A. H. Cottrell, E. Smith and K. H. Swinden, *Proc. Roy. Soc. Lond.* **A279**, 1 (1964).
12. J. P. Benthem and W. T. Koiter, *Mechanics of Fracture* (Edited by G. C. Sih), Vol. 1, p. 131. Noordhoff, Layden (1973).
13. W. R. Delameter, G. Herrmann and D. M. Barnett, *J. Appl. Mech.* **42**, 74 (1975).
14. W. R. Delameter and G. Herrmann, *Topics in Appl. Continuum Mechanics* (Edited by J. L. Zeman and F. Ziegler), p. 156. Springer-Verlag, Vienna (1974).
15. B. A. Bilby and J. B. Eshelby, *Fracture* (Edited by H. Liebowitz), Vol. 1, p. 99. Academic Press, New York (1968).
16. N. I. Muskhelishvili, *Singular Integral Equations*, (Translation editor J. R. M. Radok), p. 257. Noordhoff, Groningen (1953).